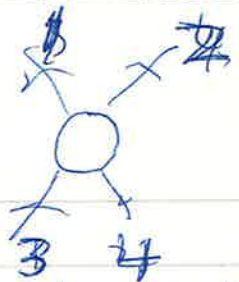

Consider Transition M.E.

$$M = \sum_i \int\int\int\int \psi_1^*(\Lambda_1)\,\psi_2^*(\Lambda_2)\,\mathcal{H}(\Lambda_1\Lambda_2\Lambda_3\Lambda_4)\,\psi_3^{(i)}(\Lambda_3)\,\psi_4^{(i)}(\Lambda_4)\,d^3\Lambda_1 d^3\Lambda_2 d^3\Lambda_3 d^3\Lambda_4 \times \delta(E_1+E_2-W)\times T$$

where $V(\Lambda_1\Lambda_2\Lambda_3\Lambda_4) = \delta(\Lambda_1-\Lambda_2)\,\delta(\Lambda_2-\Lambda_3)\,\delta(\Lambda_3-\Lambda_4)$

$$\sum_i \psi_3^{(i)}(\Lambda_3)\,\psi_4^{(i)}(\Lambda_4) = \phi(\Lambda_3-\Lambda_4)\,e^{-(\Lambda_3+\Lambda_4)^2/2\sigma}$$

$\psi_1^{(i)} = e^{i\mathbf{k}_1\cdot\Lambda_1}$, $\psi_2(\Lambda_2) = e^{i\mathbf{k}_2\cdot\Lambda_2}$

then $M = \int\int\int\int e^{-i\mathbf{k}_1\cdot\Lambda_1}\,e^{-i\mathbf{k}_2\cdot\Lambda_2}\,\delta(\Lambda_1-\Lambda_2)\,\delta(\Lambda_2-\Lambda_3)\,\delta(\Lambda_3-\Lambda_4) \times \phi(\Lambda_3-\Lambda_4)\,e^{-\frac{(\Lambda_3+\Lambda_4)^2}{2}/2\sigma}\,d\Lambda_1 d\Lambda_2 d\Lambda_3 d\Lambda_4 \times \delta(k_1^2+k_2^2-W)\times T$

$$= \int d\Lambda_1\, e^{-i(\mathbf{k}_1+\mathbf{k}_2)\cdot\Lambda_1}\,\phi(0)\,e^{-\Lambda_1^2/2\sigma}\,\delta(k_1^2+k_2^2-W)\times T$$

$$= \phi(0)\,T\,e^{-(\mathbf{k}_1+\mathbf{k}_2)^2\sigma/2}\,\delta(k_1^2+k_2^2-W)$$

So $F(\Lambda_1\Lambda_2) \int\int e^{i\mathbf{k}_1\cdot\Lambda_1}\,e^{i\mathbf{k}_2\cdot\Lambda_2}\,e^{-(\mathbf{k}_1+\mathbf{k}_2)^2\sigma}\,\delta(k_1^2+k_2^2-W)\,d\mathbf{k}_1 d\mathbf{k}_2$

with $K = k_1+k_2$, $q = k_1-k_2$, $\vec{y} = \vec{\Lambda}_1-\vec{\Lambda}_2$, $R = \Lambda_1+\Lambda_2$

so $k_1 = \frac{1}{2}(K+q)$ $k_2 = \frac{1}{2}(K-q)$

$\Lambda_1 = \frac{1}{2}(R+y)$ $\Lambda_2 = \frac{1}{2}(R-y)$

$$k_1\cdot\Lambda_1 + k_2\cdot\Lambda_2 = \frac{1}{4}\{(K+q)\cdot(R+y) + (K-q)\cdot(R-y)\}$$

$$= \frac{1}{2}(K\cdot R + q\cdot y)$$

$$F(q,R) = \iint e^{\frac{1}{2}i(k\cdot R + q\cdot y)} e^{-k^2\sigma} \delta\left(\tfrac{1}{2}(k^2+q^2) - u\right) d^3k\, d^3q.$$

$$= \iint e^{\frac{1}{2}i(kR\cos\theta_{kR} + qy\cos\theta_{qy})} e^{-k^2\sigma} \delta\left(\tfrac{1}{2}(k^2+q^2) - u\right) k^2 dk\, d\Omega_k\, q^2 dq\, d\Omega_q.$$

$$\text{Mas } \tfrac{1}{2}k^2 = u \qquad k^2 = 2u$$

$$2k\,dk = 2du \qquad k\,dk = du \qquad k^2 dk = \sqrt{2u}\,du$$

$$= \iint e^{\frac{1}{2}i(\sqrt{2u}R\cos\theta_{kR} + qy\cos\theta_{qy})} e^{-2u\sigma} \delta\left(u - \tfrac{1}{2}q^2 - w\right) \sqrt{2u}\,du\, d\Omega_R\, q^2 dq$$

$$\text{Pa } u = \tfrac{1}{2}q^2 + w$$

$$= \iint e^{\frac{1}{2}i\left(\sqrt{q^2+2w}\right)R\cos\theta_{kR} + iqy\cos\theta_{qy}} e^{-(q^2+w)\sigma} q^2 dq\, d\Omega_k\, d\Omega_q$$

$$F(y,R) = \iint e^{-k^2\sigma} \delta(k^2+q^2-w)\, e^{i(k\cdot R + q\cdot y)} d^3k\, d^3q$$

$$\text{Mas } \int e^{ik\cdot R} d\Omega_k = 2\pi \int_0^{\pi} e^{ikR\cos\theta_{kR}} \operatorname{sen}\theta\, d\theta = 2\pi \cdot \frac{1}{ikR}\left(e^{ikR} - e^{-ikR}\right) \propto \frac{\operatorname{sen} kR}{kR}$$

$$F(y,R) = \int_0^{\infty}\int_0^{\infty} e^{-k^2\sigma} \delta(k^2+q^2-w) \frac{\operatorname{sen} kR}{kR} \frac{\operatorname{sen} qy}{qy} k^2 q^2\, dk\, dq$$

$$= \frac{1}{qR} \int_0^{\infty}\int_0^{\infty} e^{-k^2\sigma} \delta(k^2+q^2-w) \operatorname{sen} kR \operatorname{sen} qy\, k\, q\, dk\, dq$$

$$q^2 = u \qquad 2q\,dq\,du$$

$$= \frac{1}{qR} \int_0^{\infty}\int_1^{\infty} e^{-k^2\sigma} \delta(u+k^2-w) \operatorname{sen} kR \operatorname{sen}\sqrt{u}\,y\, k\, dk\, du$$

$$u = w - k^2$$

$$= \frac{1}{qR} \int_0^{\infty} e^{-k^2\sigma} \operatorname{sen} kR \operatorname{sen}\left(\sqrt{w-k^2}\cdot y\right) \cdot k\, dk$$

eikR φ(q) take k=0 → φ(q) ∫= 2q₁ - h₂

$$H_{int.} = \int \psi_1^*(x)\,\psi_2^*(x)\,\psi_3(x)\,\psi_4(x)\,d^3x$$

Transition rate from $\phi_i = \int \phi(x)\,|x\rangle$ + —

is to final state $\int e^{ikx}\,|x\rangle\,dx = |k\rangle$

is $$\phi_i = \int \psi_1^+(x_1)\,\psi_2^+(x_2)\,\phi(x_1,x_2)\,dx_1 dx_2\,|0\rangle$$

Transition element is

$$\int \phi(0)\, e^{-i(k_1+k_2)\cdot x}\, d^3x\; \delta(E_1-E_2)$$
$$= \phi(0)\,\delta(k_1+k_2)\,\delta(E_1-E_2)$$

Final state $= \int |k_1\rangle\,|k_2\rangle\,\delta(k_1+k_2)\,\delta E_1-E_2 \; dk_1\,dk_2^3$

$\psi_f = \int F(k_1,k_2)\,|k_1\rangle\,|k_2\rangle\,dk_3\,d^3k_2$

$F(k_1,k_2) = \delta(k_1+k_2)\,\cancel{\delta(E_1-E_2)}\,\delta(E_1+E_2-\omega)$

$= \int F(\Omega_1,\Omega_2)\,|\Omega_1\rangle\,|k_2\rangle\,d\Omega_1\,dk_3$

$F(\Omega_1,\Omega_2) = \iint e^{ik_1\cdot\Omega_1}\, e^{ik_2\cdot\Omega_2}\,\delta(k_1+k_2)\,\delta(E_2+E_2-\omega)\; d^3k_1\,d^3k_2.$

$= \int e^{ik_2(\Omega_2-\Omega_1)}\,\delta(E_1+E_2-\omega)\,d^3k_2$

$\delta\left(\frac{k_2^2}{2m}-\omega\right)$

$= \int_0^\infty e^{i|k_2|(\Omega_2-\Omega_1)\cos\theta}\,\delta(|k_2|^2-\omega)\,|k_2|^2\,d\Omega\,dk$

$k^2 = \mu \qquad 2k\,dk = d\mu \qquad dk = \frac{d\mu}{2\sqrt{\mu}}$

$= \int_0^\infty e^{i\sqrt{\mu}|\Omega_2-\Omega_1|\cos\theta}\,\delta(\mu-\omega^2)\,\sqrt{\mu}\,d\mu\,d\Omega$

$= \int d\Omega_k\, e^{i\omega|\Omega_2-\Omega_1|\cos\theta}\,\omega$

Per $\nu = \cos\theta$ $\quad d\nu = d\cos\theta$

$$\int_{-1}^{+1} e^{ciw|r_1-r_2|\nu}\, w\, d\nu$$

$$= \frac{1}{ciw|r_1-r_2|}\left(e^{iw|r_1-r_2|} - e^{-iw|r_1-r_2|}\right)$$

$$= \frac{\sin w|r_1-r_2|}{|r_1-r_2|}$$

$$(k+q)\cdot(k+q) = (k+q)\cdot(k+q) + (k-q)\cdot(k-q)$$

$$k_1 = k+q$$
$$k_2 = k-q$$
$$r_1 = r+R$$
$$r_2 = R-r$$

$$k_1\cdot r_1 + k_2\cdot r_2 = (k+q)(r+R) + (k-q)(r-R)$$
$$= k\cdot R + q\cdot r$$

Dove $\int g(k)\, e^{ikr}\, dk$.

$$g(k) = e^{-k^2/\sigma^2}$$

$$\hookrightarrow e^{-\sigma r^2}$$

Trasformata di $\phi(0)\int e^{-i(k_1+k_2)\cdot r}\, e^{-\sigma r^2}\, d^3r\, \delta(E_1-E_2)$

$$\phi(0)\, e^{-(k_1+k_2)^2/\sigma}\, \delta(E_1+E_2-\omega)$$

$$F(k_1,k_2) = e^{-(k_1+k_2)^2/\sigma}\, \delta(k_1^2+k_2^2-\omega)$$

$$F(r_1,r_2) = \iint e^{ik_1\cdot r_1}\, e^{ik_2\cdot r_2}\, e^{-(k_1+k_2)^2/\sigma}\, \delta(k_1^2+k_2^2-\omega)\, d^3k_1\, d^3k_2$$

$$K = k_1+k_2, \quad q = k_1-k_2$$
$$r = r_1+r_2, \quad R = r_1-r_2$$

$$F(r,R) = \iint e^{-K^2/\sigma}\, \delta(K^2+q^2-\omega)\, d^3K\, d^3q \cdot e^{iK\cdot R}\, e^{iq\cdot r}$$